Exceptional sets of orthogonal projections: from classical results to current research

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Abstract

The *projection problem* of geometric measure theory is the following: given a set $A \subseteq \mathbb{R}^n$, what is the relationship between the geometric properties of A and those of its projections onto k-dimensional subspaces of $\mathbb{R}^n$? One way of answering this is to bound the dimension of the so-called *exceptional set* of orthogonal projections of A—the set of k-planes onto which the orthogonal projection of A has unusually low dimension compared to the dimension of A. Our purpose is to review the most important results of this sort, with an eye toward the technical tools involved in their proof, and to survey the author's own research in this area.

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My gratitude goes out to Bobby Wilson, a terrific adviser without whom I could never have begun nor finished this paper.

1 The projection problem

We begin with a broad overview of the projection problem with the intent of synthesizing the historical and mathematical aspects into a single narrative. The theorems we examine merit appreciation for their own sake, but they should also serve to contextualize the second section of this article, in which we specialize to an aspect of the problem that arises naturally from the preceding. While the praxis might suggest that the connections with the earlier work are tenuous, the novelty of the ideas at play should rather connote the nuanced relationship between measure and geometry that the projection problem entails. In the third section, we take a critical look at the author's contributions, including the significance of his results and their potential for future developments.

1.1 Prelude: product sets

Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$. An early result of geometric measure theory, due to Besicovitch and Moran [BM45], states the following: if A and B are Hausdorff measurable with positive and finite Hausdorff measure in their respective dimensions, then

$$\dim A + \dim B \leq \dim(A \times B), \tag{1.1}$$

where $\dim E$ denotes the *Hausdorff dimension* of a subset E of a metric space. The inspiration for the proof is that capacities behave well under projections, whence Frostman's lemma becomes a highly applicable tool. A result of Marstrand [Mar54b] extended the result to arbitrary sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$; in this setting, Frostman's lemma is unavailable, necessitating a functional analytic approach. In the same paper in which he defined the *packing dimension* $\dim_P E$, Tricot [Tri82] proved several more inequalities expressing the relationship between the dimension of a product set and those of its factors:

$$\dim(A \times B) \leq \dim A + \dim_P B \leq \dim_P(A \times B) \leq \dim_P A + \dim_P B, \tag{1.2}$$

again for arbitrary A and B subsets of Euclidean spaces. Howroyd [How95; How96], building on the work of Larman [Lar67] and Wegmann [Weg69], extended both (1.1) and (1.2) to subsets of arbitrary metric spaces.

One can interpret these as results about the behavior of dimension under projections, namely, the projections of product sets onto the factors of the ambient product space. Specifically, we have

$$A = \pi_{\mathbb{R}^n}(A \times B) \qquad \text{and} \qquad B = \dim \pi_{\mathbb{R}^m}(A \times B)$$

in the Euclidean case, where $\pi_V : W \to V$ denotes the orthogonal projection of a vector space W onto a subspace $V \subseteq W$, and

$$A = \pi_X(A \times B) \qquad \text{and} \qquad B = \pi_Y(A \times B)$$

in the general metric space setting, where the projections are potentially nonlinear. In this language, Equations (1.1) and (1.2) tell us how the Hausdorff and packing dimensions of a product set relate to those of its images under projection, and how the regularity of the images—as measured by the difference between their Hausdorff and packing dimensions—affect the strength of those relationships. In particular, if $\dim_P B = \dim B$, then $\dim A + \dim B = \dim(A \times B)$, and if $\dim_P A = \dim A$ as well, then all five expressions in (1.1) and (1.2) are equal. Peculiarly, the ansatz $\dim(A \times B) = \dim_P(A \times B)$ does not tell us anything so useful about the dimensions of the factors. One can take this as a simple expression of the ill behavior of packing dimension under projections, foreshadowing the developments of §2.

1.2 Problem setup and background material

A natural question to ask at this point is what can be said when we replace the product set $A \times B$ with an arbitrary subset of the ambient space: in view of the results in §1.1, one should hope to ascertain the size or regularity of a set from knowledge of its projections, or conversely. In the absence of two distinguished projections, one is compelled to examine projections onto *all* subspaces of $\mathbb{R}^n$ of a fixed dimension k—or at least a rich subcollection thereof—when dealing with orthogonal projections of Euclidean spaces. In the case of general metric spaces, a more novel framework is required, such as that constructed by Peres and Schlag [PS00] in their seminal paper on *generalized projections*. This level of generality exceeds our needs, but we will have occasion to discuss nonlinear projections and the fundamental notion of *transversality* in §2.

The mathematical concepts required to articulate and understand the following work do not run much deeper than the notions of Hausdorff and packing dimension and their fundamental properties, although a few other specialized concepts make appearances. The next few sections outline this foundational material.

1.2.1 Measures and dimensions

Let X be a complete separable metric space. By a ***measure*** on X we mean what is in general measure theory called an *outer measure*. Of exclusive concern to us are ***Borel regular*** measures, i.e., Borel measures μ with the following ***outer regularity*** property: for each $A \subseteq X$, there exists a Borel set $B \supseteq A$ such that $\mu(A) = \mu(B)$. A Borel regular measure μ is a ***Radon measure*** if it is ***locally finite*** in the sense that $\mu(K) < \infty$ for all $K \subseteq X$ compact. All finite Borel measures on X are (necessarily) Borel regular, hence, Radon. Given another metric space Y and a Borel function $\varphi\colon X \to Y$, we can define a Borel measure on Y by

$$\varphi_\sharp\mu(B) := \mu(\varphi^{-1}(B))$$

for all $B \subseteq Y$ and call it the ***pushforward*** of μ by φ. The equation

$$\int_Y f\, d\varphi_\sharp\mu = \int_X f \circ \varphi\, d\mu,$$

valid for all integrable functions $f : Y \to \mathbb{R}$, expresses integration with respect to $\varphi_\sharp\mu$ in term of integration with respect to μ.

Let 2^X be the power set of X and let $|F|$ be the diameter of a set $F \in 2^X$. For each $s \geq 0$, the Carathéodory construction yields a family of functions $\mathcal{H}^s_\delta\colon 2^X \to [0,\infty]$, $\delta \in (0,\infty]$, defined by

$$\mathcal{H}^s_\delta(A) := \inf\left\{\sum_{i=1}^\infty |F_i|^s\colon F_i \in 2^X,\ |F_i| \leq \delta,\ A \subseteq \bigcup_{i=1}^\infty F_i\right\}.$$

The function $\mathcal{H}^s_\infty$, called the ***s-dimensional Hausdorff content*** on X, will be of particular in §3.2. The resulting Carathéodory measure

$$\mathcal{H}^s(A) := \lim_{\delta\to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta>0} \mathcal{H}^s_\delta(A)$$

is Borel regular and is called the ***s-dimensional Hausdorff measure*** on X. For each $A \subseteq X$, there is a unique $s \in [0,\infty]$ with the following property: for all $r < s < t$, we have

$$0 = \mathcal{H}^t(A) \leq \mathcal{H}^s(A) \leq \mathcal{H}^r(A) = \infty,$$

or, equivalently,

$$s = \sup\big\{t \in [0,\infty)\colon \mathcal{H}^t(A) > 0\big\} = \inf\big\{r \in [0,\infty)\colon \mathcal{H}^r(A) = 0\big\}.$$

We write $\dim A := s$ and call this number the ***Hausdorff dimension*** of A.

Perhaps the most useful nontrivial property of Hausdorff dimension that we use is the following, as it entails that Hausdorff content is sufficient to determine Hausdorff dimension.

Proposition 1.1. *Let $A \subseteq \mathbb{R}^n$. Then $\mathcal{H}^s(A) > 0$ if and only if $\mathcal{H}^s_\infty(A) > 0$.*

Hausdorff measure—hence, Hausdorff dimension—is defined in terms of covers by *arbitrary* sets of diameter *at most* δ. It is sometimes possible to recover information about Hausdorff dimension using smaller families of covers, namely, covers by *balls* with radius *equal to* δ. For each nonempty bounded set $A \subseteq X$ and each $\delta > 0$, let

$$N(A,\delta) := \min\left\{k \in \mathbb{Z}_+\colon \exists\, x_i \in X \text{ s.t. } A \subseteq \bigcup_{i=1}^k B(x_i,\delta)\right\}, \tag{1.3}$$

where (for definiteness) we take the balls to be closed. We define the ***upper box dimension*** of A, also called its ***upper Minkowski dimension***, by

$$\begin{aligned} \overline{\dim}_B A &:= \sup\left\{t \in [0,\infty)\colon \limsup_{\delta\downarrow 0} N(A,\delta)\delta^t > 0\right\} \\ &= \inf\left\{r \in [0,\infty)\colon \limsup_{\delta\downarrow 0} N(A,\delta)\delta^r = 0\right\}. \end{aligned} \tag{1.4}$$

Comparing the admissible covers of A in the definitions of Hausdorff and upper box dimensions yields the inequality

$$\dim A \le \overline{\dim}_B A.$$

We will often deal with sets A for which equality holds.

When working with coverings of sets, quantities often arise that are unequal up to an unimportant multiplicative constant. Abstractly, if P_α and Q_α are quantities depending on some parameter α, then the notation $P_\alpha \lesssim Q_\alpha$ means that $P_\alpha \le CQ_\alpha$ for some constant C independent of α. If $P_\alpha \lesssim Q_\alpha \lesssim P_\alpha$, then we simply write $P_\alpha \sim Q_\alpha$. When several parameters are present and it is unclear from context, we use subscripts on the relations $\lesssim$ and $\sim$ to indicate on which parameters the implicit constants depend or else state these parameters explicitly.

We now turn to packing dimension, a sort of "dual" to Hausdorff dimension. It is perhaps most natural to define this in terms of *packing measure*, but, to streamline the exposition, we present an alternative characterization—the only one we will use. The ***packing dimension*** of a nonempty subset $A \subseteq X$ is given by

$$\dim_P A := \inf\left\{\sup_{i\in\mathbb{Z}_+} \overline{\dim}_B A_i\colon A = \bigcup_{i=1}^\infty A_i,\ |A_i| < \infty\right\}.$$

An important feature of packing dimension that upper box dimension lacks is ***countable stability***:

$$\dim_P \bigcup_{i=1}^\infty A_i = \sup_{i\in\mathbb{Z}_+} \dim_P A_i.$$

1.2.2 Energies of measures

This section presents tools for quantifying how "concentrated" a measure is locally. In turn, these quantities allow us to characterize the dimension of a set in terms of the measures it supports.

For $A \subseteq \mathbb{R}^n$, let $\mathcal{M}(A)$ be the family of Borel measures μ compactly supported in A and with $0 < \mu(A) < \infty$. The ***s-energy*** of a measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ is defined by

$$I_s(\mu) := \iint \|x-y\|^{-s} \, d\mu(x) \, d\mu(y),$$

and we call μ an ***s-Frostman measure*** if

$$\mu(B(x,r)) \leq r^s \qquad \forall x \in \mathbb{R}^n \quad \text{and} \quad \forall r > 0.$$

The importance of the following lemma is difficult to understate.

Theorem 1.2 (Frostman's Lemma). *Let $A \subseteq \mathbb{R}^n$ be a Borel set and $s \geq 0$. Then $\mathcal{H}^s(A) > 0$ if and only if A supports an s-Frostman measure.*

Frostman's lemma and an elementary computation with the "layer cake" formula

$$\int_0^\infty \|x-y\|^{-s} \, d\mu(y) = s \int_0^\infty \mu(B(x,r)) r^{-s-1} \, dr$$

produce the following.

Proposition 1.3. *If $\mu \in \mathcal{M}(\mathbb{R}^n)$ is s-Frostman and $s > 0$, then $I_t(\mu) < \infty$ for all $0 \leq t < s$. Conversely, if μ satisfies $I_s(\mu) < \infty$, then there exists a Borel set $B \subseteq \mathbb{R}^n$ such that $\mu \llcorner B$ (the restriction of μ to B) is s-Frostman. Consequently, if $A \subseteq \mathbb{R}^n$ is a nonempty Borel set, then*

$$\begin{aligned} \dim A &= \sup \{ s > 0 \colon A \textit{ supports an } s\textit{-Frostman measure} \} \\ &= \sup \{ s > 0 \colon \exists \mu \in \mathcal{M}(A) \textit{ s.t. } I_s(\mu) < \infty \}, \end{aligned}$$

where $\sup \varnothing := 0$.

1.2.3 Fourier analysis of measures

Again let $\mu \in \mathcal{M}(\mathbb{R}^n)$. The ***Fourier transform*** of μ is the continuous function

$$\widehat{\mu}(\xi) := \int e^{-2\pi i \xi \cdot x} \, d\mu(x),$$

$\xi \in \mathbb{R}^n$. More generally, if $T \in \mathcal{S}'(\mathbb{R}^n)$ is a tempered distribution, then we define the ***distributional Fourier transform*** of T by its action on Schwartz functions $\varphi \in \mathcal{S}(\mathbb{R}^n)$:

$$\widehat{T}(\varphi) := T(\widehat{\varphi}),$$

where

$$\widehat{\varphi}(\xi) := \int \varphi(x) e^{-2\pi i \xi \cdot x} \, dx$$

is the classical Fourier transform. If we regard a measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ as a tempered distribution, then its distributional Fourier transform agrees with its Fourier transform as a measure, so the notation is unambiguous.

For us, the utility of $\widehat{\mu}$ stems from its relationship to the s-energy of μ.

Theorem 1.4. *For every $0 < s < n$, there is a constant $\gamma(n,s) > 0$ such that*

$$I_s(\mu) = \gamma(n,s) \int |\widehat{\mu}(x)|^2 \, \|x\|^{s-n} \, dx$$

for all $\mu \in \mathcal{M}(\mathbb{R}^n)$.

This expression on the right continues to make sense even when $s \geq n$: the s-energy is necessarily infinite when $s > n$, but $\int |\widehat{\mu}(x)|^2 \, \|x\|^{s-n} \, dx$ might still be finite. As is standard in Fourier analysis (e.g., in defining fractional derivatives), we use this Fourier analytic characterization to broaden our previous definitions. For all $s \geq 0$, the ***s*-*Sobolev energy*** of a measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ is given by

$$\mathcal{I}_s(\mu) := \int |\widehat{\mu}(x)|^2 \, \|x\|^{s-n} \, dx$$

and its ***Sobolev dimension*** by

$$\dim_S \mu := \sup\,\{s > 0 \colon \mathcal{I}_s(\mu) < \infty\},$$

where $\sup \varnothing := 0$. The relation

$$\mathcal{I}_s(\mu) \sim_\mu \int |\widehat{\mu}(x)|^2 \, (1 + \|x\|)^{s-n} \, dx$$

holds for $s > 0$. It can be easier to work with a locally bounded integrand, so we typically use the characterization

$$\dim_S \mu = \sup \left\{ s > 0 \colon \int |\widehat{\mu}(x)|^2 \, (1 + \|x\|)^{s-n} \, dx < \infty \right\}$$

in proofs.

1.2.4 The Grassmannian

Let $\mathbf{Gr}(n,k)$ be the ***Grassmannian***—the manifold whose elements are the k-dimensional linear subspaces of $\mathbb{R}^n$. The Grassmannian carries the structure of a metric space with metric $d(V,W) := \|\pi_V - \pi_W\|$, where $\pi_V \colon \mathbb{R}^n \to V \subseteq \mathbb{R}^n$ and $\pi_W \colon \mathbb{R}^n \to W \subseteq \mathbb{R}^n$ are orthogonal projections and $\|\cdot\|$ is the operator norm induced by the Euclidean norm. It also enjoys an invariant measure induced by the action of the orthogonal group $\mathrm{O}(n)$: if θ_n is the Haar measure on $\mathrm{O}(n)$ and $V \in \mathbf{Gr}(n,k)$, then

$$\gamma_{n,k}(A) := \theta_n\big(\{T \in \mathrm{O}(n) \colon T(V) \in A\}\big),$$

where the definition does not depend on the specific choice of V. This Radon measure is ***uniformly distributed*** in the sense that $\gamma_{n,k}(B(V,r)) = \gamma_{n,k}(B(W,r))$ for all $V, W \in \mathbf{Gr}(n,k)$ and all $r > 0$, and it is ***$k(n-k)$-Ahlfors regular*** in the sense that $\gamma_{n,k}(B(V,r)) \sim r^{k(n-k)}$ for $r \in (0,1]$.

A key estimate used in proving projection theorems is the following.

Lemma 1.5. *If $\nu \in \mathcal{M}(\mathbf{Gr}(n,k))$ is p-Frostman, then*

$$\nu\big(\{V \in \mathbf{Gr}(n,k) \colon \|\pi_V(x)\| \leq \delta\}\big) \lesssim_n \big(\delta/\|x\|\big)^{p-k(n-k-1)}$$

and

$$\nu\big(\{V \in \mathbf{Gr}(n,k) \colon \mathrm{dist}(x,V) \leq \delta\}\big) \lesssim_n \big(\delta/\|x\|\big)^{p-(k-1)(n-k)}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$ and all $0 < \delta < \infty$.

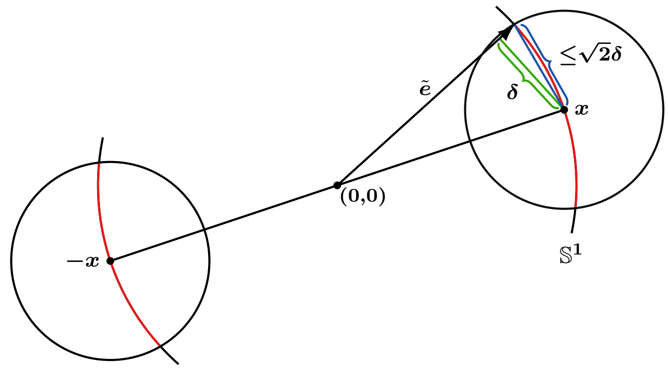

Figure 1. The red arcs represent the set of $e \in \mathbb{S}^1$ such that $\operatorname{dist}(x, \operatorname{span} e) \leq \delta$, where by scaling invariance we take $\|x\| = 1$ without loss of generality. The arcs lie in a pair of Euclidean balls of radius $\|x - \tilde{e}\| \leq \sqrt{2}\delta$ centered at $\pm x$, so their total ν-measure is at most $2(\sqrt{2}\delta)^p \sim \delta^p$ when ν is p-Frostman.

Its proof is an elementary geometric argument whose crux is to estimate the number of balls in a $\delta/\|x\|$-cover of the set $\{V \in \mathbf{Gr}(n,k) : x \in V\}$. We can identify $\mathbf{Gr}(2,1)$ with the unit circle $\mathbb{S}^1$, and in this case the proof is particularly simple; see Figure 1.

Occasion will arise in the proof of Proposition 1.12 to apply Frostman's lemma to subsets of $\mathbb{S}^1$ (and, more generally, in the proof of Theorem 1.15 to subsets of $\mathbf{Gr}(n,k)$). This is valid in view of Theorem 1.2 because these spaces admit isometric embeddings into $\mathbb{R}^m$ for sufficiently large m, and Lemma 1.5 will be applied to the Frostman measures on these spaces to obtain invaluable integral estimates.

1.2.5 Symbolic dynamics

A family $(f_i)_{i=1}^k$ of contractions of $\mathbb{R}^n$ is called an ***iterated function system*** (***IFS***). It is a fact that there exists a unique nonempty compact set $K \subset \mathbb{R}^n$ that is invariant under the system, i.e., such that

$$\bigcup_{i=1}^{k} f_i(K) = K;$$

this is called the ***limit set*** or ***attractor*** of the IFS. In the event that the sets $f_i(K)$ are pairwise disjoint, we say that the IFS satisfies the ***strong separation condition*** (***SSC***). If the f_i are all conformal maps, we call K ***self-conformal***; if they are similarities (i.e., if $f_i(x) = a_i x + b_i$, $a_i \in \mathbb{R}$, $b_i \in \mathbb{R}^n$), we call K ***self-similar***. The middle-thirds Cantor set in $\mathbb{R}$ and the four-corner Cantor set in $\mathbb{R}^2$ are perhaps the most popular examples of self-similar sets, and both are generated by IFS satisfying the SSC.

What follows is closely adapted from [Ram02], and presenting the main result therein is our reason for laying down this notation. In particular, these definitions will help us to articulate what it means for a parametrized family of IFS to be "degenerate" for some values of the parameter.

Let $(f_i)_{i=1}^k$ be an IFS on $\mathbb{R}^n$. Here and in §2.1, we assume that the f_i are $C^{1,\beta}$ for some $\beta > 0$; that they are conformal; and that the *local contraction ratios* $|\det Df_i|^{1/n}$ are uniformly bounded.

The set $\Sigma := \{1, ..., k\}^{\mathbb{Z}_+}$ of all (infinite) sequences of integers 1 through k is called the ***symbol space*** of the IFS. Given $\omega = (\omega_1, \omega_2, ...) \in \Sigma$, we denote $\omega^m := (\omega_1, ..., \omega_m)$ and $f_{\omega^m} := f_{\omega_1} \circ \cdots \circ f_{\omega_m}$. We moreover define

$$f_\omega(x) := \lim_{m \to \infty} f_{\omega^m}(x);$$

the limit always exists by the Cantor intersection theorem, as the f_i are all contractions. Lastly, we define a projection map $\Pi \colon \Sigma \to \mathbb{R}^n$ by

$$\Pi(\omega) := f_\omega(0).$$

When working with families of IFS indexed by a parameter t, the subscripted symbol Π_t will denote the corresponding projection.

1.2.6 Homogeneous sets and dimension conservation

The following summarizes some essential material from Furstenberg's paper [Fur08]. It is hardly an overstatement to call this the backbone of §2.2 and §3.2.

A Lipschitz function $f \colon \mathbb{R}^n \to \mathbb{R}^m$ is said to be ***dimension conserving*** (***DC***) for a set $A \subseteq \mathbb{R}^n$ if there exists $\Delta \geq 0$ such that

$$\Delta + \dim \left\{ y \in \mathbb{R}^m \colon \dim(f^{-1}(y) \cap A) \geq \Delta \right\} \geq \dim A, \tag{1.5}$$

where $\dim \varnothing := -\infty$. Heuristically, f is DC for A if a substantial portion of the dimension lost by A under f is accounted for by the dimension of the fibers: it is a sort of "rank-nullity inequality." The pathological Example 7.8 of [Fal14] shows that even the projections of a product set onto the coordinate axes may radically fail to be DC for that set.

The ***Hausdorff metric*** on the class $\mathcal{K}$ of nonempty compacta in $\mathbb{R}^n$ is defined by

$$\rho(H, K) := \inf \{\varepsilon \geq 0 \colon H \subseteq K_\varepsilon \text{ and } K \subseteq H_\varepsilon\} = \inf \{\varepsilon \geq 0 \colon H \cup K \subseteq H_\varepsilon \cap K_\varepsilon\},$$

$H, K \in \mathcal{K}$, where A_ε is the closed ε-neighborhood of A. With the Hausdorff metric, $\mathcal{K}$ is a complete metric space.

We now describe the sets $K \in \mathcal{K}$ with which we will be working. Scaling and translating a set does not affect the dimension of the projection of a set in any direction, so we assume without loss of generality that $K \subseteq [0, 1]^n$. A closed set $K' \subseteq [0, 1]^n$ is called a ***miniset*** of K if there exists an expanding homothety $\varphi(x) = rx + b$ ($|r| \geq 1$) such that $K' \subseteq \varphi(K)$. A closed set $K'' \subseteq [0, 1]^n$ is called a ***microset*** of K if there exists a sequence $(K'_j)_{j=1}^\infty$ of minisets of K such that $\left(K'_j \cap [0, 1]^n\right)_{j=1}^\infty$ converges to K'' in the Hausdorff metric: $\rho(K'_j, K'') \to 0$. Finally, K is said to be ***homogeneous*** if all its microsets are minisets; that is, if the class of minisets of K is closed in $\mathcal{K}$.

Loosely, K is homogeneous if it looks the same at arbitrarily small scales: even if the minisets K'_j must be contained in larger and larger expansions of K as $j \to \infty$ (meaning they resemble smaller and smaller subsets of K), there still exists a scale on which the limiting set K'' coincides with a subset of K at that scale. Besides the embedded submanifolds of $\mathbb{R}^n$, the concrete examples to bear in mind are the self-similar sets containing no rotations and satisfying the SSC. One non-example is the set $\left\{\frac{1}{j}\right\}_{j=1}^\infty \cup \{0\}$: the interval $[0, 1]$ is a miniset but not a microset.

Appreciation for the definition of "dimension conserving" is essential for understanding §3. On the other hand, the technical definition of a "homogeneous set" is less important, as the only two properties we will require are the following (cf. [Fur08] p. 407 and Theorem 6.2).

Proposition 1.6. *If K is homogeneous, then* $\dim K = \overline{\dim}_B K$.

Theorem 1.7. *If $K \subset \mathbb{R}^n$ is homogeneous and $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then f is DC for K. In particular, every projection map is DC for K.*

Theorem 1.7 is highly nontrivial and its proof is steeped in ergodic theory. The connections between geometric measure theory and ergodic theory run rich, deep, and (regrettably) outside the scope of this paper.

1.3 The Besicovitch-Federer and Marstrand projection theorems

Historically, the first major theorem concerning the projection problem for planar sets came in [Bes39]. The final result in a groundbreaking trio of papers, it was asserted by Besicovitch to have stature comparable to the characterization of rectifiable and purely unrectifiable sets in terms of the a.e. existence of tangents. Call an $\mathcal{H}^s$-measurable set $A \subseteq \mathbb{R}^n$ an ***s-set*** if $0 < \mathcal{H}^s(A) < \infty$, and for each $e \in \mathbb{S}^1$, let $\pi_e : \mathbb{R}^2 \to \mathbb{R}$ denote the orthogonal projection $\pi_e(x) := e \cdot x$ and $A_e := \pi_e(A)$.

Proposition 1.8. *Let $A \subset \mathbb{R}^2$ be a 1-set.*

- **(a)** *A is purely unrectifiable if and only if $\mathcal{L}^1(A_e) = 0$ for $\mathcal{H}^1$-a.e. $e \in \mathbb{S}^1$.*
- **(b)** *A has a countably rectifiable subset of positive $\mathcal{H}^1$-measure if and only if $\mathcal{L}^1(A_e) > 0$ for $\mathcal{H}^1$-a.e. $e \in \mathbb{S}^1$.*

An elementary geometric argument reveals that a countably rectifiable 1-set projects onto a set of measure 0 in at most one direction, and this simple observation allows us to sharpen Proposition 1.8 in an extreme way.

Corollary 1.9. *Let $A \subset \mathbb{R}^2$ be a 1-set.*

- **(a)** *A is purely unrectifiable if and only if $\mathcal{L}^1(A_e) = 0$ for at least two distinct $e \in \mathbb{S}^1$.*
- **(b)** *If A has a countably rectifiable subset of positive $\mathcal{H}^1$-measure, then $\mathcal{L}^1(A_e) = 0$ for at most one $e \in \mathbb{S}^1$.*

Federer [Fed47] extended Proposition 1.8 to higher dimensions as follows. Hereafter, $\pi_V : \mathbb{R}^n \to \mathbb{R}^k \cong V$ denotes the orthogonal projection onto the k-dimensional subspace $V \subset \mathbb{R}^n$, identified with $\mathbb{R}^k$, and we notate $A_V := \pi_V(A)$ for $A \subseteq \mathbb{R}^n$.

Theorem 1.10 (Besicovitch-Federer Projection Theorem). *Let $A \subset \mathbb{R}^n$ be a k-set.*

- **(a)** *A is purely unrectifiable if and only if $\mathcal{L}^k(A_V) = 0$ for $\gamma_{n,k}$-a.e. $V \in \mathbf{Gr}(n,k)$.*
- **(b)** *A has a countably rectifiable subset of positive $\mathcal{H}^k$-measure if and only if $\mathcal{L}^k(A_V) > 0$ for $\gamma_{n,k}$-a.e. $V \in \mathbf{Gr}(n,k)$.*

In fact, Federer proved something more general and abstract, but even a superficial treatment would take us too far afield from the theory in which our primary interest lies. See, however, Theorem 3.2.27 of [Fed69], which pairs with Theorem 1.10 to give a straightforward analogue of Corollary 1.9.

Following the results relating projections and regularity, Marstrand [Mar54a] finally made headway on the problem of extending the dimension results of §1.1 above, i.e., of relating the dimension of a set to the dimensions of its images under projection. Interestingly, Marstrand's aforementioned paper [Mar54b] on product sets in $\mathbb{R}^n$ was published *later* that same year, but this should not come as a surprise when one considers the *ad hoc* methods required to work in the total absence of regularity hypotheses.

That is not to say that [Mar54a] depends heavily on methods of a general nature: on the contrary, Marstrand's work, like that of Besicovitch, tended to be highly geometric—hence the confinement to planar sets in [Mar54a]. His main theorem on projections is the following. Modern proofs forgo delicate constructions in favor of more widely applicable tools and techniques, and the remainder of this section concerns the relationship between the potential theoretic and Fourier transform methods for proving Marstrand-type projection theorems.

Proposition 1.11. *Let $A \subseteq \mathbb{R}^2$ be Borel.*

(a) *If $\dim A \leq 1$, then $\dim A_e = \dim A$ for a.e. $e \in \mathbb{S}^1$.*
(b) *If $\dim A > 1$, then $\mathcal{L}^1(A_e) > 0$ for a.e. $e \in \mathbb{S}^1$.*

In [Kau68], Kaufman strengthened Proposition 1.11(a) through a capacity theoretic argument and reproved (b) via Fourier analysis. Although ahistorical, we state and prove Kaufman's strengthening of (a) and a later variant of (b) due to Falconer [Fal82], as they bring to the fore *exceptional sets*, our primary object of study.

Proposition 1.12. *Let $A \subseteq \mathbb{R}^2$ be Borel.*

(a) *If $0 \leq s \leq \dim A \leq 1$, then*
$$\dim \{e \in \mathbb{S}^1 \colon \dim A_e < s\} \leq s. \tag{1.6}$$
(b) *If $0 \leq s \leq 1 < \dim A \leq s+1$, then*
$$\dim \{e \in \mathbb{S}^1 \colon \dim A_e < s\} \leq 1 - (\dim A - s). \tag{1.7}$$

We begin with a lemma about the regularity of the exceptional sets.

Lemma 1.13. *Let $K \subseteq \mathbb{R}^n$ be compact and $s < \min\{\dim K, k\}$. Then the exceptional set $\{V \in \mathbf{Gr}(n,k) \colon \dim K_V < s\}$ is $G_{\delta\sigma}$ and, in particular, Borel.*

Proof. Using Proposition 1.1, write
$$\begin{aligned} E &:= \{V \in \mathbf{Gr}(n,k) \colon \dim K_V < s\} \\ &= \bigcup_{\substack{j \in \mathbb{Z}_+ \\ j > s^{-1}}} \left\{V \in \mathbf{Gr}(n,k) \colon \mathcal{H}^{s-j^{-1}}_\infty(K_V) = 0\right\} \\ &= \bigcup_{\substack{j \in \mathbb{Z}_+ \\ j > s^{-1}}} \bigcap_{i \in \mathbb{Z}_+} \left\{V \in \mathbf{Gr}(n,k) \colon \mathcal{H}^{s-j^{-1}}_\infty(K_V) < i^{-1}\right\}. \end{aligned} \tag{1.8}$$

Given $t := s - j^{-1}$ and $V \in \mathbf{Gr}(n,k)$ such that $\mathcal{H}^t_\infty(K_V) < c$ for some $c > 0$, there exists a cover $\{U_\ell\}_{\ell=1}^m$ of K_V by open sets such that $\sum_{\ell=1}^m |U_\ell|^t < c$. Since K is compact (and so are its projections), $\{U_\ell\}_{\ell=1}^m$ also covers K_W for all $W \in \mathbf{Gr}(n,k)$ sufficiently close to V, namely, for

$$\|\pi_V - \pi_W\| < \operatorname{dist}\left(K, \mathbb{R}^n \setminus \bigcup_{\ell=1}^m U_\ell\right).$$

In particular, $\mathcal{H}^t_\infty(K_W) < c$ for all W in a neighborhood of V, so the map $V \mapsto \mathcal{H}^t_\infty(K_V)$ is upper semicontinuous. The final line of (1.8) is therefore a countable union of countable intersections of open sets, i.e., a $G_{\delta\sigma}$ set. ■

Proof of Proposition 1.12(a). There is nothing to prove if $\dim A = 0$ or $s = 0$, so assume that $0 < s \le \dim A$. (The hypothesis that $\dim A \le 1$ is not needed, but part (b) is always stronger when $\dim A > 1$.) In addition, since A can be approximated from within by compact sets, we may assume that A is compact so that the lemma above applies.

Suppose for a contradiction that $\mathcal{H}^s(E_s) > 0$, where

$$E_s := \{e \in \mathbb{S}^1 : \dim A_e < s\}.$$

Then, by Frostman's lemma, there exists $\nu \in \mathcal{M}(E_s)$ such that $\nu(B(e,r)) \le r^s$ for all $e \in \mathbb{S}^1$ and $r > 0$. Similarly, letting $0 < t < s$ and applying Frostman's lemma on A gives $\mu \in \mathcal{M}(A)$ satisfying $\mu(B(x,r)) \le r^t$ for all $x \in \mathbb{R}^2$ and $r > 0$. Letting $\mu_e := \pi_{e\sharp}\mu$, we claim that $I_t(\mu_e) < \infty$ for ν-a.e. $e \in \mathbb{S}^1$. To see how this yields a contradiction, notice that $\mu_e \in \mathcal{M}(A_e)$, so we will have $\dim A_e \ge t$ for ν-a.e. e by Proposition 1.3. Taking a sequence $t = t_j \uparrow s$ then gives $\dim A_e \ge s$ for ν-a.e. e, contradicting the definition of E_s.

The idea is to integrate the t-energies with respect to ν and show that the result is finite. Begin with the following estimate: by the "layer cake" formula and Lemma 1.5,

$$\begin{aligned}
&\int_{\mathbb{S}^1} |\pi_e(x) - \pi_e(y)|^{-t}\, d\nu(e) = \int_0^\infty \nu\big(\{e \in \mathbb{S}^1 : |\pi_e(x-y)|^{-t} \ge \alpha\}\big)\, d\alpha \\
&\quad = \int_0^{\|x-y\|^{-t}} \nu\big(\{e \in \mathbb{S}^1 : |\pi_e(x-y)|^{-t} \ge \alpha\}\big)\, d\alpha + \int_{\|x-y\|^{-t}}^\infty \nu\big(\{e \in \mathbb{S}^1 : |\pi_e(x-y)| \le \alpha^{-1/t}\}\big)\, d\alpha \\
&\quad \lesssim \nu(\mathbb{S}^1)\|x-y\|^{-t} + \int_{\|x-y\|^{-t}}^\infty \big(\alpha^{-1/t}/\|x-y\|\big)^s\, d\alpha \\
&\quad = \nu(\mathbb{S}^1)\|x-y\|^{-t} + \|x-y\|^{-s} \int_{\|x-y\|^{-t}}^\infty \alpha^{-s/t}\, d\alpha \lesssim \|x-y\|^{-t}.
\end{aligned}$$

It then follows by Tonelli's theorem and the computation above that

$$\begin{aligned}
\int_{\mathbb{S}^1} I_t(\mu_e)\, d\nu(e) &= \int_{\mathbb{S}^1} \left[\iint \|x-y\|^{-t}\, d\mu_e(x)\, d\mu_e(y)\right] d\nu(e) \\
&= \int_{\mathbb{S}^1} \left[\iint |\pi_e(x) - \pi_e(y)|^{-t}\, d\mu(x)\, d\mu(y)\right] d\nu(e) \\
&= \iint \left[\int_{\mathbb{S}^1} |\pi_e(x) - \pi_e(y)|^{-t}\, d\nu(e)\right] d\mu(x)\, d\mu(y) \\
&\lesssim \iint \|x-y\|^{-t}\, d\mu(x)\, d\mu(y) = I_t(\mu) < \infty,
\end{aligned}$$

since μ is s-Frostman and $s > t$. Because the integral of $I_t(\mu_e)$ with respect to $d\nu(e)$ is finite, we must have $I_t(\mu_e) < \infty$ for ν-a.e. $e \in \mathbb{S}^1$. Therefore, $\mathcal{H}^s(E_s) = 0$ and, in particular, $\dim E_s \le s$. ■

A heuristic one gleans from this proof is that the concentration of a pushforward measure μ_e typically reflects the concentration of μ itself, subject to the constraint imposed by the dimension of the space onto which we are projecting. When projecting a set onto a subspace of lower dimension—in the planar case, when $\dim A > 1$—this caveat becomes significant, as the projected measures μ_e behave in an "at most 1-dimensional fashion." We would like to say that they typically behave in a "$(\dim A)$-dimensional fashion," but this is impossible from a classical perspective: a nonzero Borel measure λ on $\mathbb{R}^k$ cannot satisfy $\lambda(B(x,r)) \leq r^\gamma$ for all $x \in \mathbb{R}^k$ and $r > 0$ when $\gamma > k$.

The solution discovered by Kaufman [Kau68] and augmented by Falconer [Fal82] uses the Fourier transform to recast questions about the density of μ_e on $\operatorname{span} e$ as questions about the concentration of $\widehat{\mu_e}$ near $\operatorname{span} e^\perp$.

Proof of Proposition 1.12(b). We show the following:

Let $\mu \in \mathcal{M}(\mathbb{R}^2)$ and $0 < s < \dim_S \mu$. Then

$$\dim \{e \in \mathbb{S}^1 \colon \dim_S \mu_e < s\} \leq \max\{1 - (\dim_S \mu - s), 0\}. \tag{1.9}$$

To see that this suffices, let $\mu \in \mathcal{M}(A)$ be such that $s < \dim_S \mu \leq \dim A \leq s + 1$. By Proposition 1.3, Theorem 1.4, and the definition of Sobolev dimension,

$$\{e \in \mathbb{S}^1 \colon \dim A < s\} \subseteq \{e \in \mathbb{S}^1 \colon \dim_S \mu_e < s\},$$

so

$$\dim \{e \in \mathbb{S}^1 \colon \dim A < s\} \leq 1 - (\dim_S \mu - s)$$

provided (1.9) holds. Taking the infimum of the right-hand side over all $\mu \in \mathcal{M}(A)$ then gives (1.7).

Let $E_s := \{e \in \mathbb{S}^1 : \dim_S \mu_e < s\}$—a Borel set by the same argument as in the proof of Lemma 1.13—and suppose for a contradiction that (1.9) does not hold. Then, taking

$$\dim E_s > p > 1 - (\dim_S \mu - s),$$

we obtain by Frostman's lemma a p-Frostman measure $\nu \in \mathcal{M}(E_s)$. We claim that

$$\int_{\mathbb{S}^1} \int_{\mathbb{R}} |\widehat{\mu_e}(\eta)|^2 \, (1 + |\eta|)^{s-1} \, d\eta \, d\nu(e) < \infty, \tag{1.10}$$

and that this yields the desired contradiction. Indeed, this inequality implies that $\mathcal{I}_s(\mu_e) < \infty$ for ν-a.e. $e \in \mathbb{S}^1$. An application of Fatou's lemma yields $\dim_S \mu_e < s$ (a *strict* inequality) whenever $\mathcal{I}_s(\mu_e) < \infty$, whence $\nu(E_s) = 0$, contradicting the fact that $\nu \in \mathcal{M}(E_s)$.

Owing to the equation $\widehat{\mu_e}(\eta) = \widehat{\mu}(\eta e)$, (1.10) would follow immediately from the finiteness of $I_s(\mu)$ if ν were the surface measure (i.e., arc length) on $\mathbb{S}^1$. The workaround involves introducing a Schwartz function to "take" the ηe from $\widehat{\mu}$ and leave it with a single variable u, furnishing us a factor of $|\widehat{\mu}(u)|^2$ that we integrate with respect to Lebesgue measure on $\mathbb{R}^2$. If the Schwartz function decays sufficiently rapidly, this yields an estimate on the integral in (1.10) in terms of the energy of μ.

Let $\varphi \in \mathcal{S}(\mathbb{R}^2)$ satisfy $\varphi|_{\operatorname{spt}\mu} \equiv 1$, so that $\mu = \varphi\mu$ and, in turn, $\widehat{\mu} = \widehat{\varphi\mu} = \widehat{\varphi} * \widehat{\mu}$. Then, by the Cauchy-Schwarz inequality,

$$|\widehat{\mu}(\xi)|^2 = |(\widehat{\varphi} * \widehat{\mu})(\xi)|^2 = \left| \int \widehat{\varphi}(\xi - u)\widehat{\mu}(u) \, du \right|^2 = \left| \int \widehat{\varphi}(\xi - u)^{1/2} \Big(\widehat{\varphi}(\xi - u)^{1/2} \widehat{\mu}(u) \Big) du \right|^2$$

$$\leq \int |\widehat{\varphi}(u)|\, du \int |\widehat{\varphi}(\xi - u)||\widehat{\varphi}(u)|^2\, du \lesssim_{\varphi} \int |\widehat{\varphi}(\xi - u)||\widehat{\mu}(u)|^2\, du,$$

whence the bound

$$\begin{aligned}
\int_{\mathbb{S}^1}\int_{\mathbb{R}} |\widehat{\mu_e}(\eta)|^2 (1+|\eta|)^{s-1}\, d\eta\, d\nu(e) &= \int_{\mathbb{S}^1}\int_{\mathbb{R}} |\widehat{\mu}(\eta e)|^2 \,(1+|\eta|)^{s-1}\, d\eta\, d\nu(e) \\
&\lesssim \int_{\mathbb{S}^1}\int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} |\widehat{\varphi}(\eta e - u)||\widehat{\mu}(u)|^2\, du\right)(1+|\eta|)^{s-1}\, d\eta\, d\nu(e) \\
&= \int_{\mathbb{R}^2} |\widehat{\mu}(u)|^2 \left(\int_{\mathbb{S}^1}\int_{\mathbb{R}} |\widehat{\varphi}(\eta e - u)|(1+|\eta|)^{s-1}\, d\eta\, d\nu(e)\right) du \\
&\lesssim \int_{\mathbb{R}^2} |\widehat{\mu}(u)|^2 \left(\int_{\mathbb{S}^1}\int_{\mathbb{R}} (1+\|\eta e - u\|)^{-N}(1+|\eta|)^{s-1}\, d\eta\, d\nu(e)\right) du
\end{aligned}$$

follows for all $N \in \mathbb{N}$, where Tonelli's theorem justifies the third line and the Schwartz class bound on $\widehat{\varphi}(\eta e - u)$ justifies the fourth.

If we can show that the inner integral in the last line satisfies

$$\int_{\mathbb{S}^1}\int_{\mathbb{R}} (1+\|\eta e - u\|)^{-N}(1+|\eta|)^{s-1}\, d\eta\, d\nu(e) \lesssim (1+\|u\|)^{s-1-p} \tag{1.11}$$

for some N, then (1.10) will follow from the previous equation block:

$$\int_{\mathbb{S}^1}\int_{\mathbb{R}} |\widehat{\mu_e}(\eta)|^2 \,(1+|\eta|)^{s-1}\, d\eta\, d\nu(e) \lesssim \int_{\mathbb{R}^2} |\widehat{\mu}(u)|^2 (1+\|u\|)^{s-1-p}\, du = I_{(s-1-p)+2}(\mu) < \infty,$$

per our choice of p satisfying $s - p + 1 < \dim_S \mu$.

In particular, let $N > \max\{1+p, s\}$. To complete the proof, we estimate (1.11) via a dyadic decomposition of $\mathbb{S}^1 \times \mathbb{R}$ into annuli centered at u. While daunting in appearance, the computations are elementary. Denote

$$A_0 := \left\{(e,\eta) \in \mathbb{S}^1 \times \mathbb{R} \colon \|\eta e - u\| \leq 2^{-1}\right\}$$

and, for $i \geq 1$,

$$A_i := \left\{(e,\eta) \in \mathbb{S}^1 \times \mathbb{R} \colon 2^{i-2} < \|\eta e - u\| \leq 2^{i-1}\right\}.$$

We partition the domain of integration into a disc and two families of annuli, all centered at u:

$$\begin{aligned}
&\int_{\mathbb{S}^1}\int_{\mathbb{R}} (1+\|\eta e - u\|)^{-N}(1+|\eta|)^{s-1}\, d\eta\, d\nu(e) \\
&\quad = \sum_{i=0}^{\infty} \iint_{A_i} (1+\|\eta e - u\|)^{-N}(1+|\eta|)^{s-1}\, d\eta\, d\nu(e) \\
&\quad = \iint_{A_0} (1+\|\eta e - u\|)^{-N}(1+|\eta|)^{s-1}\, d\eta\, d\nu(e) \\
&\qquad + \sum_{\{i \in \mathbb{Z}_+ \colon \|u\| > 2^i\}} \iint_{A_i} (1+\|\eta e - u\|)^{-N}(1+|\eta|)^{s-1}\, d\eta\, d\nu(e) \\
&\qquad + \sum_{\{i \in \mathbb{Z}_+ \colon \|u\| \leq 2^i\}} \iint_{A_i} (1+\|\eta e - u\|)^{-N}(1+|\eta|)^{s-1}\, d\eta\, d\nu(e).
\end{aligned} \tag{1.12}$$

Let us estimate each of the three terms separately. First, the integral over the disc: notice that $|\eta| \lesssim \|u\|$ on A_0, we can pull out the $(1+|\eta|)^{s-1}$ as $(1+\|u\|)^{s-1}$. Using also the domain inclusion

$$A_0 \subseteq \big\{(e,\eta) \in \mathbb{S}^1 \times \mathbb{R} \colon \operatorname{dist}(u, \operatorname{span} e) \le 2^{-1}\big\},$$

the trivial bound

$$\int_{\mathbb{R}} (1+\|\eta e - u\|)^{-N}\, d\eta = \int_{\mathbb{R}} (1+|\eta|)^{-N}\, d\eta \lesssim 1,$$

and the paramount Lemma 1.5, we get

$$\begin{aligned}
&\iint_{A_0} (1+\|\eta e - u\|)^{-N}(1+|\eta|)^{s-1}\, d\eta\, d\nu(e) \\
&\qquad \lesssim (1+\|u\|)^{s-1} \int_{\{e\in\mathbb{S}^1 \colon \operatorname{dist}(u,\operatorname{span} e)\le 2^{-1}\}} \int_{\mathbb{R}} (1+\|\eta e - u\|)^{-N}\, d\eta\, d\nu(e) \\
&\qquad \lesssim (1+\|u\|)^{s-1}\, \nu\big(\big\{e \in \mathbb{S}^1 \colon \operatorname{dist}(u,\operatorname{span} e) \le 2^{-1}\big\}\big) \\
&\qquad \lesssim (1+\|u\|)^{s-1-p}.
\end{aligned} \tag{1.13}$$

The first family of dyadic annuli A_i consists of those such that the disc bounded by $\partial A_i \setminus \partial A_{i-1}$ does not contain the origin, and they satisfy the inclusion

$$A_i \subseteq \big\{(e,\eta) \in \mathbb{S}^1 \times [-2^i, 2^i] \colon \operatorname{dist}(u, \operatorname{span} e) \le 2^i\big\}. \tag{1.14}$$

As such, we again have $|\eta| \lesssim \|u\|$, plus Lemma 1.5 estimate on the integral with respect to $d\nu(e)$, with combine with the above domain inclusion to yield

$$\begin{aligned}
&\sum_{\{i\in\mathbb{Z}_+ \colon \|u\|\le 2^i\}} \iint_{A_i} (1+\|\eta e - u\|)^{-N}(1+|\eta|)^{s-1}\, d\eta\, d\nu(e) \\
&\qquad \lesssim (1+\|u\|)^{s-1} \sum_{\{i\in\mathbb{Z}_+ \colon \|u\|> 2^i\}} \iint_{A_i} (1+\|\eta e - u\|)^{-N}\, d\eta\, d\nu(e) \\
&\qquad \lesssim (1+\|u\|)^{s-1} \sum_{\{i\in\mathbb{Z}_+ \colon \|u\|> 2^i\}} 2^i \cdot 2^{-iN}\, \nu\big(\big\{e\in\mathbb{S}^1 \colon \operatorname{dist}(u,\operatorname{span} e) \le 2^i\big\}\big) \\
&\qquad \lesssim (1+\|u\|)^{s-1} \sum_{i=1}^{\infty} 2^{(1-N)i} \big(2^i/\|u\|\big)^p \\
&\qquad \lesssim (1+\|u\|)^{s-1-p} \sum_{i=1}^{\infty} 2^{(1-N+p)i} \\
&\qquad \lesssim (1+\|u\|)^{s-1-p},
\end{aligned} \tag{1.15}$$

per our choice of $N > p+1$. Lastly, on the second family of dyadic annuli A_i—those with inner radii greater than $\|u\|$—we use the previous inclusion (1.14) and estimate $(1+\|\eta e - u\|)^{-N} \lesssim 2^{-iN}$, integrate with respect to $d\eta$ for $|\eta| \le 2^i$; and bound the integral with respect to $d\nu(e)$ by $\nu(\mathbb{S}^1) \lesssim 1$:

$$\begin{aligned}
&\sum_{\{i\in\mathbb{Z}_+ \colon \|u\|\le 2^i\}} \iint_{A_i} (1+\|\eta e - u\|)^{-N}(1+|\eta|)^{s-1}\, d\eta\, d\nu(e) \\
&\qquad \lesssim \sum_{\{i\in\mathbb{Z}_+ \colon \|u\|\le 2^i\}} 2^{-iN} \int_{\mathbb{S}^1} \int_{|\eta|\le 2^i} (1+|\eta|)^{s-1}\, d\eta\, d\nu(e) \\
&\qquad \lesssim \sum_{\{i\in\mathbb{Z}_+ \colon \|u\|\le 2^i\}} 2^{-iN} \int_{|\eta|\le 2^i} (1+|\eta|)^{s-1}\, d\eta
\end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{\{i\in\mathbb{Z}_+ :\, \|u\|\le 2^i\}} 2^{-iN}\cdot 2^{(i+1)(s-1)} = \sum_{\{i\in\mathbb{Z}_+ :\, \|u\|\le 2^i\}} 2^{(s-N)i+(s-1)} \\
&\lesssim 1 \lesssim (1+\|u\|)^{s-1-p},
\end{aligned} \tag{1.16}$$

since $N > s$ and all u of interest belong to the compact set $\operatorname{spt}\mu$. Combining (1.13), (1.15), and (1.16) with (1.12) gives (1.11), as intended. ■

While the proof of (b) is unquestionably more complicated than the proof of (a), the two are, as noted above, similar in spirit. In fact, as was done in the proof of (b), the statement of (a) can be rephrased in terms of the Sobolev dimensions of measures:

Let $\mu \in \mathcal{M}(\mathbb{R}^2)$ and $0 < s < \dim_S \mu$. Then

$$\dim\{e \in \mathbb{S}^1 \colon \dim_S \mu_e < s\} \le s.$$

Peres and Schlag [PS00] formulated their theory of generalized projections in this same language—a testament its widespread applicability.

The higher-dimensional analogues of Propositions 1.11 and 1.12 are straightforward to state. The proof of Theorem 1.15—the "quantitative" Marstrand projection theorem—is hardly different from that of Proposition 1.12 above, and we hope that, by proving only this planar case, we bring the working principles to the fore.

Theorem 1.14 (Marstrand Projection Theorem). *Let $A \subseteq \mathbb{R}^n$ be Borel.*

- **(a)** *If $\dim A \le k$, then $\dim A_V = \dim A$ for $\gamma_{n,k}$-a.e. $V \in \mathbf{Gr}(n,k)$.*
- **(b)** *If $\dim A > k$, then $\mathcal{L}^k(A_V) > 0$ for $\gamma_{n,k}$-a.e. $V \in \mathbf{Gr}(n,k)$.*

Theorem 1.15. *Let $A \subseteq \mathbb{R}^n$ be Borel.*

- **(a)** *If $0 \le s \le \dim A \le k$, then*
$$\dim\{V \in \mathbf{Gr}(n,k) \colon \dim A_V < s\} \le k(n-k) - (k-s).$$
- **(b)** *If $0 \le s \le k \le \dim A \le k(n-k)+s$, then*
$$\dim\{V \in \mathbf{Gr}(n,k) \colon \dim A_V < s\} \le k(n-k) - (\dim A - s).$$

As an aside, the direct proof of Theorem 1.14 differs little from that of Theorem 1.15: where one integrates with respect to a measure supported on the exceptional set in the latter, one instead integrates with respect to $\gamma_{n,k}$ in the former. (As remarked in the proof of (b), this leads to vast simplification, but the simplicity is more practical than theoretical.) However, a simple argument using the continuity of measure from below also shows that Theorem 1.15 implies Theorem 1.14.

The dimension bound given in Theorem 1.15(a) has two interpretations: first, as the dimension of $\mathbf{Gr}(n,k)$ minus the minimum dimension loss; and second, when written in the equivalent form $k(n-k-1)+s$, as the dimension of the set $\{V \in \mathbf{Gr}(n,k) : \pi_V(x) = 0\}$ plus the maximum dimension of the images. This first interpretation applies equally well to part (b), but the second does not. An important consequence of this observation is that the existence of a proof of (b) using potential theory alone is dubious, since part (a) hinges on the first inequality of Lemma 1.5—stated in terms of where $\pi_V(x)$ is small—whereas (b) utilizes the second inequality of that lemma. This also suggests the appropriate modifications of the proof of (b) required to recover (a).

The possibility of doing so attests to the robustness of the Fourier transform in solving geometric measure theory problems as opposed to, e.g., problems in partial differential equations, where only the linear theory is amenable to the Fourier transform.

We have said a great deal about the Marstrand-type theorems and little about those of the Besicovitch-Federer type. Part of the reason hinges on the fundamental differences between the study of sets of integer and fractional dimension. The decomposability of sets of integer dimension into a countably rectifiable and a purely unrectifiable part—the one of which has a countable cover by Lipschitz graphs, density 1 a.e., and a unique tangent a.e., the other of which has antithetical properties—affords a panoply of power tools in their study. For example, projection theorems for rectifiable sets reduce to statements about projections of Lipschitz graphs, and those for unrectifiable sets benefit from the Lebesgue density theorem and its relatives, of which there are no fractional-dimensional analogues.

Another reason is our emphasis on exceptional sets, where again the contrast is stark: Corollary 1.9 states that the exceptional set in the Besicovitch-Federer projection theorem is—for rectifiable planar sets—at most a singleton, whereas the extension of Proposition 1.11 to 1.12 entails substantial nuance for comparatively little improvement.

2 Packing dimension of exceptional sets

While Marstrand's projection theorem is the gold standard for results concerning the dimensions of projections, part (a) is only known to be sharp when $s = \dim A$. In fact, the following theorem of Oberlin [Obe12] for planar sets—also conjectured to be true in higher dimensions—shows that Proposition 1.12 is *never* sharp for Borel sets of codimension at least 1 when s is sufficiently small.

Theorem 2.1. *For every Borel set $A \subseteq \mathbb{R}^2$ with $\dim A \leq 1$,*

$$\dim \left\{e \in \mathbb{S}^1 \colon \dim A_V < \tfrac{1}{2} \dim A\right\} = 0.$$

See [Mat15] §5.4 for more on the current state of this problem.

Whether or not Marstrand's projection theorem is sharp in any general circumstances, the question of what special cases yield sharper bounds remains lucrative, and will likely remain so even after loose ends regarding the former are sealed. The remainder of this paper addresses the following instantiation of this question: for what families of projections and for what sorts of sets can we obtain interesting bounds on the *packing* dimension of the exceptional set of projections?

Owing to the general inequality $\dim A \leq \dim_P A$, packing dimension estimates are inherently stronger than Hausdorff dimension estimates, *ceteris paribus*. However, an example of Orponen [Orp15] shows that a naïve packing dimension analogue of Proposition 1.12 is impossible.

Proposition 2.2. *There exists a compact set $K \subset \mathbb{R}^2$ with $\mathcal{H}^1(K) > 0$ and a dense G_δ set $E \subset \mathbb{S}^1$ such that $\dim K_e = 0$ for all $e \in E$. In particular,*

$$\dim_P \{e \in \mathbb{S}^1 \colon \dim K_e = 0\} = 1.$$

As such, packing dimension bounds on exceptional sets demand greater nuance. Given the disparity of the methods required to work with packing and Hausdorff dimensions, one might fear that no salient connection with Marstrand's theorem exists at all. The two papers examined in this section, Rams [Ram02] and Orponen [Orp15], refute this concern, as the packing dimension bounds they obtain under their respective hypotheses coincide with the Hausdorff dimension bounds of Theorem 1.15.

2.1 Rams' theorem

Theorem 1.1 of Rams [Ram02] implies a result similar to that of Kaufman, but with a packing dimension bound replacing the Hausdorff dimension bound on the exceptional set; with this additional strength comes the drawback that the set under consideration must be fairly regular. We begin by stating his theorem in full. The terminology and notation are substantial, so the reader may wish to view §2 and Definition 4.4 of [Ram02] (in addition to §1.2.5 of this paper) for additional background.

Theorem 2.3 (Rams' Theorem). *Let $V \subset \mathbb{R}^n$ be a bounded open set, and for each $t \in \overline{V}$, let $\big(f_i(\,\cdot\,;t)\big)_{i=1}^N$ be a conformal IFS on $\mathbb{R}^n$ with limit set K_t. Assume that each f_i is $C^{1,\beta}$ in all n variables and n parameters for some $\beta > 0$, and denote by $\sigma(t)$ the solution to Bowen's equation*

$$P(\sigma(t)\chi_t) = 0,$$

where P is the topological pressure and χ_t is the Lyapunov exponent of the IFS. Lastly, for each $s \geq 0$, let G_s be the exceptional set

$$G_s := \{t \in \overline{V} \colon \dim K_t \leq s\}.$$

If $\big(f_i(\,\cdot\,;t)\big)_{i=1}^N$ satisfies the transversality condition, then, for all $t \in V$,

$$\limsup_{r\to 0} \dim_P(G_s \cap B_r(t)) \leq s \qquad \textit{for all} \qquad 0 \leq s < \min\{n, \sigma(t)\}.$$

The level of generality exceeds our needs, so we state the following corollary.

Corollary 2.4. *Let $V \subset \mathbb{R}^n$ be a bounded open set, and for each $t \in \overline{V}$, let $\big(f_i(\,\cdot\,;t)\big)_{i=1}^N$ be a family of similarities on $\mathbb{R}^n$ with limit set K_t. Assume that each f_i is smooth in all n variables and n parameters, and denote by $\sigma(t)$ the solution to Moran's equation*

$$\sum_{i=1}^N a_i(t)^{\sigma(t)} = 0,$$

where $a_i(t) \in (-1, 1)$ is the similarity ratio of $f_i(\,\cdot\,;t)$. If $\big(f_i(\,\cdot\,;t)\big)_{i=1}^N$ satisfies the transversality condition, then

$$\dim_P \{u \in \overline{V} \colon \dim K_u \leq s\} \leq s \qquad \textit{for all} \qquad 0 \leq s < \min\left\{ n, \sup_{t\in\overline{V}} \sigma(t) \right\}.$$

For a given t, the quantity $\sigma(t)$ is called the ***similarity dimension*** of the IFS. A question of independent interest is when the similarity dimension of an IFS equals the Hausdorff, upper box, or packing dimension of its attractor. A simple covering argument shows that the SSC implies the equality of all these quantities.

Let $\rho_e : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ denote the orthogonal projection onto the hyperplane orthogonal to the vector $e \in \mathbb{S}^{n-1}$, and suppose $K \subset \mathbb{R}^n$ is the limit set of an IFS $(g_i)_{i=1}^N$ that satisfies the SSC. For each $e \in \mathbb{S}^{n-1}$, we can choose a section ρ_e^{-1} of ρ_e and define $f_i(\,\cdot\,; e) := \rho_e \circ g_i \circ \rho_e^{-1} \colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$. Then each $(f_i(\,\cdot\,; e))_{i=1}^N$ is an IFS on $\mathbb{R}^{n-1}$ with limit set $\rho_e(K)$. As such, the problem of determining the exceptional set of projections for K is equivalent to determining the exceptional set of the IFS $(f_i(\,\cdot\,; e))_{i=1}^N$.

This setup allows for an application of Rams' theorem to obtain the following.

Proposition 2.5. *Let $K \subset \mathbb{R}^n$ be the limit set of a family of similarities containing no rotations or reflections and satisfying the SSC. Then*

$$\dim_P \{e \in \mathbb{S}^{n-1} \colon \dim \rho_e(K) \le s\} \le s \qquad \text{for all} \qquad 0 \le s < \min\{n-1, \dim K\}.$$

Our main result will subsume this as a special case; nevertheless, we include its proof to shed light on the relationship between his work and our own, and to give a sense of just how strong Rams' transversality condition is.

Proof. It suffices to work in local coordinates, so we let $V \subset \mathbb{S}^{n-1}$ be an open set whose closure $\overline{V}$ is diffeomorphic to a bounded subset of $\mathbb{R}^{n-1}$. These local coordinates also afford us a consistent identification of the tangent hyperplanes to $\mathbb{S}^{n-1}$ with $\mathbb{R}^{n-1}$. Dispensing with these technicalities, we we simply refer to our parameter space as $\mathbb{S}^{n-1}$ and use the formula $\rho_e(x) = x - (x \cdot e)e$ for the orthogonal projections.

[Step 1] Let $(g_i)_{i=1}^N$ be an IFS on $\mathbb{R}^n$ with limit set K and satisfying the SSC. We seek to produce a smooth family of IFS on $\mathbb{R}^{n-1}$ to which we can apply Corollary 2.4.

To this end, we define $\big(f_i(\,\cdot\,; e)\big)_{i=1}^N$ by

$$f_i(\xi; e) := (\rho_e \circ g_i)\big(\rho_e^{-1}(\xi)\big)$$

for each $e \in \mathbb{S}^{n-1}$, where $\rho_e^{-1}(\xi)$ is any preimage of the point $\xi \in \mathbb{R}^{n-1}$. This definition is unambiguous because g_i takes the form $g_i(x) = a_i x + b_i$ for some $a_i \in \mathbb{R}$ and $b_i \in \mathbb{R}^n$, whence

$$(\rho_e \circ g_i)\big(\rho_e^{-1}(\xi)\big) = \rho_e\big(a_i \rho_e^{-1}(\xi) + b_i\big) = a_i \rho_e\big(\rho_e^{-1}(\xi)\big) + \rho_e(b_i) = a_i \xi + \rho_e(b_i)$$

for any choice of $\rho_e^{-1}(\xi)$. This also shows that $\big(f_i(\,\cdot\,; e)\big)_{i=1}^N$ is smooth in both x and e, as $\rho_e(b_i) = b_i - (b_i \cdot e)e$.

[Step 2] We show that $\big(f_i(\,\cdot\,; e)\big)_{i=1}^N$ is a transverse family. Let $\omega, \kappa \in \Sigma$, where $\omega_1 \ne \kappa_1$, and let $f_{\omega^m}(\xi; e)$ denote the composite map $f_{\omega_1}(\,\cdot\,; e) \circ \cdots \circ f_{\omega_m}(\,\cdot\,; e)$ evaluated at ξ. Then

$$\begin{aligned} f_{\omega^m}(\xi; e) &= \big((\rho_e \circ g_{\omega_1} \circ \rho_e^{-1}) \circ \cdots \circ (\rho_e \circ g_{\omega_m} \circ \rho_e^{-1})\big)(\xi) \\ &= \big(\rho_e \circ (g_{\omega_1} \circ \cdots \circ g_{\omega_m}) \circ \rho_e^{-1}\big)(\xi) \\ &= (\rho_e \circ g_{\omega^m} \circ \rho_e^{-1})(\xi) \end{aligned}$$

for any section ρ_e^{-1} of ρ_e. Therefore, by the continuity of ρ_e,

$$f_\omega(\xi; e) := \lim_{m \to \infty} f_{\omega^m}(\xi; e) = \rho_e\Big(\lim_{m \to \infty} g_{\omega^m}\Big)\big(\rho_e^{-1}(\xi)\big) = (\rho_e \circ g_\omega)\big(\rho_e^{-1}(\xi)\big).$$

In particular, we can take $\rho_e^{-1}(0) = 0$, so

$$\Pi_e(\omega) = f_\omega(0; e) = (\rho_e \circ g_\omega)(0) = \rho_e(\Pi(\omega));$$

likewise for κ.

Denote $z = (z_1, ..., z_n) = \Pi(\omega) - \Pi(\kappa)$, and suppose

$$|\rho_u(z)| = \big|\rho_u(\Pi(\omega)) - \rho_u(\Pi(\kappa))\big| = |\Pi_u(\omega) - \Pi_u(\kappa)| < 2^{-1}c \tag{2.1}$$

for some $u \in \mathbb{S}^{n-1}$, where $c \in (0,1]$ is a constant such that $\operatorname{dist}(g_i(K), g_j(K)) > c$ for all $i \neq j$. Such an c exists because $(g_i)_{i=1}^N$ satisfies the SSC. Since $\Pi(\omega) \in g_{\omega_1}(K)$, $\Pi(\kappa) \in g_{\kappa_1}(K)$, and $\omega_1 \neq \kappa_1$, it follows that $|z| > c$—a fact we shall use shortly.

To show transversality, we must compute

$$\det D_e\big(\rho_e(z)\big)\big|_{e=u}.$$

The determinant is invariant under a linear change of coordinates, so we can rotate our coordinate system so that $u = e_n = (0, ..., 0, 1)$. Consider $h : e \mapsto \rho_e(z)$ as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$, i.e., by extending $\rho_e(z)$ to take parameter values in $\mathbb{R}^n$. Considered as an $n \times n$ matrix, the jth column of the derivative $D_e h(e)|_{e=e_n}$ is given by the directional derivative

$$\begin{aligned} \frac{d}{dr}\rho_{e_n + re_j}(z)\big|_{r=0} &= \frac{d}{dr}\big(z - (z\cdot(e_n + re_j))(e_n + re_j)\big)\big|_{r=0} \\ &= -z_j(e_n + re_j) - (z_n + rz_j)e_j\big|_{r=0} = -z_j e_n - z_n e_j, \end{aligned}$$

yielding

$$D_e h(e)|_{e=e_n} = \begin{pmatrix} -z_n & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -z_n & 0 \\ -z_1 & -z_2 & \cdots & -2z_n \end{pmatrix}.$$

Since $D_e h(e)|_{e=e_n}$ restricts to an automorphism of the tangent plane $(\operatorname{span} e_n)^\perp \cong T_{e_n}\mathbb{S}^{n-1}$, and since the standard coordinate frame $(e_1, ..., e_n)$ is adapted to $\mathbb{S}^{n-1}$ at the north pole e_n, the matrix of this restricted linear map is obtained simply by omitting the nth row and nth column of the matrix. That is, $D_e\big(\rho_e(z)\big)\big|_{e=e_n} = -z_n I_{n-1}$ and, consequently,

$$\det D_e\big(\rho_e(z)\big)\big|_{e=e_n} = \det\big(-z_n I_{n-1}\big) = (-z_n)^{n-1}.$$

Now, since $z = \rho_{e_n}(z) + z_n e_n$, $|\rho_{e_n}(z)|^2 < 2^{-1}c^2$, and $|z|^2 > c^2$, it must be that $|z_n|^2 = |z_n e_n|^2 > 2^{-1}c^2$ and, in turn,

$$\Big|D_e\big(\rho_e(z)\big)\big|_{e=e_n}\Big| = |z_n|^{n-1} > 2^{-(n-1)/2}c^{n-1} \geq 2^{-1}c.$$

In view of Equation (2.1), we conclude that, whenever $\omega_1 \neq \kappa_1$,

$$|\Pi_u(\omega) - \Pi_u(\kappa)| < 2^{-1}c \quad \text{implies} \quad \Big|\det D_e\big(\Pi_e(\omega) - \Pi_e(\kappa)\big)\big|_{e=u}\Big| = \Big|\det D_e\big(\rho_e(z)\big)\big|_{e=u}\Big| > 2^{-1}c,$$

so $\big(f_i(\,\cdot\,; e)\big)_{i=1}^N$ satisfies the transversality condition.

[Step 3] We apply Corollary 2.4 to get

$$\dim_P \{e \in \mathbb{S}^{n-1} : \dim \rho_e(K) \leq s\} \leq s \qquad \text{for all} \qquad 0 \leq s < \min\Big\{n-1, \sup_{e\in\mathbb{S}^{n-1}} \sigma(e)\Big\}.$$

Since the given IFS consists only of similarities $g_i(x) = a_i x + b_i$, the similarity dimension $\sigma(e)$ of each IFS is the same as the similarity dimension of $(g_i)_{i=1}^N$, namely, $\dim K$, so the proposition is proved. ■

2.2 A theorem of Orponen

While Rams' theorem is very general in that it applies to families of nonlinear maps, the proof of Proposition 2.5 reveals two weaknesses that greatly limit its scope. First, the conformal family of IFS on $\mathbb{R}^n$ must depend on exactly n parameters (and in a nontrivial way). This is why the proposition only applies to projections onto hyperplanes. Second, the family of projections must satisfy a strong transversality condition that even the classical examples of IFS—for example, the similarities generating the Sierpiński triangle—do not enjoy; hence our requirement that the IFS satisfy the SSC. The second critique in some sense generalizes the first, owing to Rams' strict definition of transversality, but his definition does readily extend to m-parameter families on $\mathbb{R}^n$ for $m \neq n$.

For planar sets, Orponen's result in [Orp15] allows us to forgo the rotation and separation conditions of Proposition 2.5 in the case that K is self-similar, or to instead assume that K is homogeneous.

Proposition 2.6. *Let $K \subset \mathbb{R}^2$ be homogeneous or self-similar. Then*

$$\dim_P \{e \in \mathbb{S}^1 \colon \dim K_e \leq s\} \leq s$$

for all $0 \leq s < \dim K$.

While this is interesting simply in view of Proposition 2.5, the search for packing dimension bounds on $\{e \in \mathbb{S}^1 : \dim K_e \leq s\}$—the set of directions in which the *Hausdorff* dimension is small—has something of historical significance. Hausdorff dimension plays several distinct roles in Theorems 1.14 and 1.15: in measuring the size of the set A, in measuring the sizes of its projections A_V, and in measuring the sizes of the exceptional sets E_s. The first Marstrand-type theorem to incorporate packing dimension, originally due to Järvenpää [Jär94] and subsequently sharpened by Falconer and Howroyd [FH97], is the analogue of Theorem 1.15 with $\dim_P A$ and $\dim_P A_V$ replacing $\dim A$ and $\dim A_V$, respectively. In particular, Theorem 13 and Proposition 18 of [FH97] combine to give the following:

Theorem 2.7. *Let $A \subseteq \mathbb{R}^n$ be a Borel set and $0 < s \leq k \leq n$. Then*

$$\dim \left\{ V \in \mathbf{Gr}(n,k) \colon \dim_P A_V < \frac{\dim_P A}{1 + (s^{-1} - n^{-1}) \dim_P A} \right\} \leq k(n-k) - (k-s).$$

If each instance of packing dimension were replaced with Hausdorff, then the resulting statement would be weaker than Theorem 1.15, and [Jär94] gives an example showing that nothing stronger can be said in general. This is another manifestation of the poor behavior of packing dimension under projections observed in the prelude.

In view of Proposition 2.2 and Theorem 2.7, the only natural combination of dimensions left to try is to bound the packing dimension of the set of directions in which $\dim_P A$ is small. Orponen does this for planar sets in [Orp15] (cf. Theorem 1.6), but those results defy comparison to Proposition 1.12. In order to identify circumstances in which we can truly obtain a sharper bound than that given by Proposition 1.12 (or, more generally, by Theorem 1.15), one must look at the packing dimension (or any dimension Dim satisfying $\dim A \leq \operatorname{Dim} A$) of the exceptional set but the Hausdorff dimension of the projections. Proposition 2.2 demands that this endeavor only be taken for sets for sets A in a much smaller class than the class of Borel sets.

3 Higher dimensions and further directions

This final section details the author's completed and continuing work building on and extending the work of Rams, Furstenberg, and Orponen. His most significant result of this research to date is the following higher dimensional analogue of Proposition 2.6.

Theorem 3.1. *If $K \subset \mathbb{R}^n$ is a homogeneous set or a self-similar with finite rotation group, then*

$$\dim_P \{V \in \mathbf{Gr}(n,k) \colon \dim K_V \leq s\} \leq k(n-k) - (k-s) \tag{3.1}$$

for all $0 \leq s < \dim K$.

Our proof will closely follow Orponen's: it is primarily a geometric combinatorial argument, cleverly paired with some rudimentary properties of homogeneous sets.

3.1 Counting points on $\mathbf{Gr}(n,k)$

Counting arguments pervade the literature on packing dimension, and our main theoretical tool in this capacity is a discrete version of Lemma 1.5. The motivating question is this: given $x, y \in \mathbb{R}^n$, a δ-separated set $E \subset \mathbf{Gr}(n,k)$, and $c > 0$, for how many $V \in E$ do we have $\|\pi_V(x) - \pi_V(y)\| \leq c\delta$? The following—conceivably of independent interest—provides a broad answer.

Lemma 3.2. *Let $x \in \mathbb{R}^n \setminus \{0\}$ and $0 < \delta_1, \delta_2 < \infty$, and let $E \subset \mathbf{Gr}(n,k)$ be δ_2-separated. Then*

$$\operatorname{card}\big\{V \in E \colon \|\pi_V(x)\| \leq \delta_1\big\} \lesssim_{n,k} \delta_1^k \, \delta_2^{-k(n-k)} \|x\|^{-k}.$$

Proof. Since $r^{k(n-k)} \lesssim \gamma_{n,k}(B(V,r))$ for all $V \in \mathbf{Gr}(n,k)$ and $r \in (0,1]$, it follows from the separation hypothesis on E that

$$\delta_2^{k(n-k)} \operatorname{card}\big\{V \in E \colon \|\pi_V(x)\| \leq \delta_1\big\} \lesssim \gamma_{n,k}\big(\big\{V \in \mathbf{Gr}(n,k) \colon \|\pi_V(x)\| \leq \delta_1\big\}\big).$$

We likewise have $\gamma_{n,k}(B(V,r)) \lesssim r^{k(n-k)}$, whence

$$\gamma_{n,k}\big(\big\{V \in \mathbf{Gr}(n,k) \colon \|\pi_V(x)\| \leq \delta_1\big\}\big) \lesssim \delta_1^k \|x\|^{-k}$$

by Lemma 1.5. Combining the above two inequalities and dividing through by $\delta_2^{k(n-k)}$ completes the proof. ■

The question at the start of this section is answered by replacing x with $x - y$ and applying the linearity of π_V.

3.2 Proof of Theorem 3.1

Given a set $K \subseteq \mathbb{R}^n$, we denote by $\Delta(V)$ the set of all $\Delta \geq 0$ such that the dimension conservation condition (1.5) holds with $A = K$ and $f = \pi_V$:

$$\Delta + \dim\big\{y \in \mathbb{R}^m \colon \dim(\pi_V^{-1}(y) \cap K) \geq \Delta\big\} \geq \dim K.$$

Theorem 3.1 will follow readily from the following two lemmas. The first is essentially true by definition, but it formalizes the idea that dimension conservation allows one to recover information about the dimension of a set from knowledge of the dimension of a projection.

Lemma 3.3. *If $K \subset \mathbb{R}^n$ is homogeneous, then*

$$\big\{V \in \mathbf{Gr}(n,k) \colon \dim K_V \leq s\big\} \subseteq \big\{V \in \mathbf{Gr}(n,k) \colon \Delta \geq \dim K - s \ \forall \Delta \in \Delta(V)\big\}. \tag{3.2}$$

Proof. Since $K \subset \mathbb{R}^n$ is homogeneous, π_V is DC for K for all $V \in \mathbf{Gr}(n,k)$; hence, $\Delta(V) \neq \varnothing$. Suppose $\dim K_V \leq s$. If $\Delta \in \Delta(V)$, then, by the definition of dimension conservation,

$$\Delta + \dim \left\{ y \in \mathbb{R}^k \colon \dim\bigl(K \cap \pi_V^{-1}(y)\bigr) \geq \Delta \right\} \geq \dim K.$$

Of course,

$$K_V \supseteq \left\{ y \in \mathbb{R}^k \colon \dim\bigl(K \cap \pi_V^{-1}(y)\bigr) \geq \Delta \right\},$$

for if $y \notin K_V$, then $\dim\bigl(K \cap \pi_V^{-1}(y)\bigr) = \dim \varnothing = -\infty < \Delta$. Therefore, by the monotonicity of dimension,

$$\Delta + s \geq \Delta + \dim K_V \geq \dim K. \qquad \blacksquare$$

The second lemma is much more involved, and the reader is encouraged to skip ahead to see how it is used before examining its proof.

Lemma 3.4. *Let $K \subset \mathbb{R}^n$ be a compact set with $\dim K = \overline{\dim}_B K$. Then*

$$\dim_P \left\{ V \in \mathbf{Gr}(n,k) \colon \exists \Delta \in \Delta(V) \textit{ s.t. } \Delta \geq \dim K - s \right\} \leq k(n-k) - (k-s) \tag{3.3}$$

for all $0 \leq s < \dim K$.

Proof. The $s = 0$ case follows from the $s > 0$ case by letting $s \downarrow 0$, so we assume that $s > 0$.

[Step 1] Let $\gamma := \dim K$, and let E denote the exceptional set on the left-hand side of Equation (3.3). We begin by making a reduction that affords us the small parameters required for our argument. In particular, we claim that it suffices to show the following for all $\varepsilon > 0$ sufficiently small and for all $0 < \tau < \gamma - s$:

$$\overline{\dim}_B E_{\varepsilon,\tau} \leq k(n-k) - (k-s) + 3\tau, \tag{3.4}$$

where

$$E_{\varepsilon,\tau} := \left\{ V \in \mathbf{Gr}(n,k) \colon \exists \Delta \geq \gamma - s \text{ s.t. } \mathcal{H}_\infty^{\gamma-\Delta-\tau}\bigl(\left\{ y \in \mathbb{R}^k \colon \mathcal{H}_\infty^{\Delta-\tau}\bigl(K \cap \pi_V^{-1}(y)\bigr) > \varepsilon \right\}\bigr) > \varepsilon \right\}.$$

To prove the validity of this reduction, suppose that $\Delta \geq \gamma - s$ for some $\Delta \in \Delta(V)$. (In particular, π_V is DC for K.) Then, for every $\tau > 0$,

$$\dim \left\{ y \in \mathbb{R}^k \colon \dim\bigl(K \cap \pi_V^{-1}(y)\bigr) \geq \Delta \right\} > \gamma - \Delta - \tau.$$

Consequently, if $0 < \tau < \gamma - s \leq \Delta$, the set on the left-hand side of this inequality has infinite $(\gamma - \Delta - \tau)$-dimensional Hausdorff measure and, consequently, positive $(\gamma - \Delta - \tau)$-dimensional Hausdorff content:

$$\mathcal{H}_\infty^{\gamma-\Delta-\tau}\bigl(\left\{ y \in \mathbb{R}^k \colon \dim\bigl(K \cap \pi_V^{-1}(y)\bigr) \geq \Delta \right\}\bigr) > 0.$$

Similarly, $\dim\bigl(K \cap \pi_V^{-1}(y)\bigr) \geq \Delta$ implies $\mathcal{H}_\infty^{\Delta-\tau}\bigl(K \cap \pi_V^{-1}(y)\bigr) > 0$, so

$$\mathcal{H}_\infty^{\gamma-\Delta-\tau}\bigl(\left\{ y \in \mathbb{R}^k \colon \mathcal{H}_\infty^{\Delta-\tau}\bigl(K \cap \pi_V^{-1}(y)\bigr) > 0 \right\}\bigr) > 0.$$

Writing

$$\left\{ y \in \mathbb{R}^k \colon \mathcal{H}_\infty^{\Delta-\tau}\bigl(K \cap \pi_V^{-1}(y)\bigr) > 0 \right\} = \bigcup_{m \in \mathbb{Z}_+} \left\{ y \in \mathbb{R}^k \colon \mathcal{H}_\infty^{\Delta-\tau}\bigl(K \cap \pi_V^{-1}(y)\bigr) > m^{-1} \right\},$$

we can by countable additivity find $\varepsilon = m^{-1} > 0$ such that

$$\mathcal{H}^{\gamma-\Delta-\tau}_\infty\big(\big\{y \in \mathbb{R}^k : \mathcal{H}^{\Delta-\tau}_\infty\big(K \cap \pi_V^{-1}(y)\big) > \varepsilon\big\}\big) > 0.$$

The left-hand side is decreasing in ε, so we can further reduce ε if necessary to obtain

$$\mathcal{H}^{\gamma-\Delta-\tau}_\infty\big(\big\{y \in \mathbb{R}^k : \mathcal{H}^{\Delta-\tau}_\infty\big(K \cap \pi_V^{-1}(y)\big) > \varepsilon\big\}\big) > \varepsilon.$$

Hence, $V \in E_{\varepsilon,\tau}$, whence we can write

$$E \subseteq \bigcup_{m=N}^{\infty} E_{m^{-1},\tau}.$$

for any $N \in \mathbb{Z}_+$. It follows from our definition of packing dimension that (3.4) implies (3.3), so we set out to prove (3.4).

[Step 2] Let $\varepsilon > 0$ and $0 < \tau < \gamma - s$. We discretize the problem and define a family of relations, indexed by $V \in \mathbf{Gr}(n,k)$, that relate distant points $x, y \in K$ whose projections $\pi_V(x), \pi_V(y)$ are close.

Let $\gamma' > \gamma$, $d := (\gamma - s - \tau)^{-1}$, and

$$\delta < \eta := \frac{\varepsilon^d}{n^{1/2}\, 2^{2+d}\, (2^{n-k}+1)^d}.$$

(The significance of this requirement on δ will become apparent later.) By the definition of upper box dimension, there exists a finite subset $K_0 \subseteq K$ such that $\operatorname{card} K_0 \lesssim \delta^{-\gamma'}$ and

$$K \subseteq \bigcup_{x \in K_0} B(x, \delta).$$

For each $V \in \mathbf{Gr}(n,k)$, let $\mathcal{T}_V$ be the family of *δ-fat $(n-k)$-planes* of the following form:

$$\pi_V^{-1}\left(\prod_{i=1}^{k} \big[j_i\delta, (j_i+1)\delta\big)\right) \subset \mathbb{R}^n,$$

$j_1, \dots, j_k \in \mathbb{Z}$. (For succinctness, we will call these "elements of $\mathcal{T}_V$" or "fat planes.") These are half-open neighborhoods of fibers of π_V over points of the lattice $\big(\mathbb{Z} + \frac{1}{2}\big)^k\delta$, and their disjoint union is all of $\mathbb{R}^n$. We define relations $\sim_V$ on $\mathbb{R}^n$ by

$$\begin{aligned} x \sim_V y \quad \iff \quad & \|x-y\| > 2\eta = \frac{\varepsilon^d}{2^{1+d}(2^{n-k}+1)^d} \quad \text{and} \quad \exists\, T \in \mathcal{T}_V \quad \text{s.t.} \\ & B(x,\delta) \cap T \neq \varnothing \quad \text{and} \quad B(y,\delta) \cap T \neq \varnothing. \end{aligned} \tag{3.5}$$

This states that $\sim_V$ relates points of $\mathbb{R}^n$ that are not too close to each other, but that nevertheless belong to the same fat plane, adjacent fat planes, or fat planes with a common neighboring fat plane. In particular, although the points are fairly distant from each other, their projections onto V are quite close.

[Step 3] Let $E_0 \subseteq E_{\varepsilon,\tau}$ be any δ-separated subset, and define the energy of E_0 by

$$\mathcal{E} := \sum_{V \in E_0} \operatorname{card}\big\{(x,y) \in K_0^2 : x \sim_V y\big\}. \tag{3.6}$$

We use this energy to bound $\operatorname{card} E_0$ and, in turn, $\overline{\dim}_B E_{\varepsilon,\tau}$.

To obtain an upper bound, note that, given $x, y \in K_0$, the number of k-planes $V \in E_0$ such that $x \sim_V y$ is $\lesssim \delta^{-k(n-k)+k} \|x-y\|^{-k}$ by Lemma 3.2. Hence, for a fixed $x \in K_0$, we have

$$\sum_{V \in E_0} \operatorname{card}\left\{y \in K_0 \colon x \sim_V y\right\} \lesssim \delta^{-k(n-k)+k} \left(\max_{\substack{y \in K_0, V \in E_0 \colon \\ x \sim_V y}} \|x-y\|^{-k} \right) \operatorname{card}\left\{y \in K_0 \colon \|x-y\| > 2\eta\right\}$$
$$\leq \delta^{-k(n-k)+k} (2\eta)^{-k} \operatorname{card} K_0 \lesssim \delta^{-k(n-k)+k-\gamma'},$$

where both $\lesssim$ indicate inequality up to a constant depending only on n, γ, s, ε, and τ, but not δ. Summing over all $x \in K_0$ gives

$$\mathcal{E} \lesssim \delta^{-k(n-k)+k-\gamma'} \operatorname{card} K_0 \lesssim \delta^{-k(n-k)+k-2\gamma'}.$$

To place a lower bound on $\mathcal{E}$, we estimate the individual terms in the sum (3.6). Let $V \in E_0 \subseteq E_{\varepsilon,\tau}$ and $\Delta \in \Delta(V)$. Unwinding the definition of $E_{\varepsilon,\tau}$, we see that there exist $j \gtrsim \delta^{\Delta+\tau-\gamma}$ fat planes $T_i \in \mathcal{T}_V$ and points $y_i \in \pi_V(T_i)$, $i = 1, ..., j$, with the following property: if $W_i := \pi_V^{-1}(y_i)$ denotes the $(n-k)$-plane contained in T_i that "passes through" y_i, then

$$\mathcal{H}^{\Delta-\tau}_{\infty}(K \cap W_i) > \varepsilon. \tag{3.7}$$

To ensure that we are counting "enough" of the relations $x \sim_V y$ that hold on K_0, we checkerboard each $(n-k)$-plane $T_i \in \mathcal{T}_V$ with boxes or "checkerboard squares"

$$R = T_i \cap \pi_{V^\perp}^{-1} \left(\prod_{\ell=1}^{n-k} \left[4 i_\ell \eta, 4(i_\ell+1)\eta\right) \right),$$

$i_1, ..., i_{n-k} \in \mathbb{Z}$ (see Figure 2). Recalling that we chose $\delta < \eta$, we see that

$$|R| = \left(\sum_{\ell=1}^{n-k} (4\eta)^2 + \sum_{\ell=n-k+1}^{n} \delta^2 \right)^{1/2} < \left(\sum_{\ell=1}^{n} (4\eta)^2 \right)^{1/2} = n^{1/2} \cdot 4\eta = \frac{\varepsilon^d}{2^d (2^{n-k}+1)^d}$$

and, consequently, that

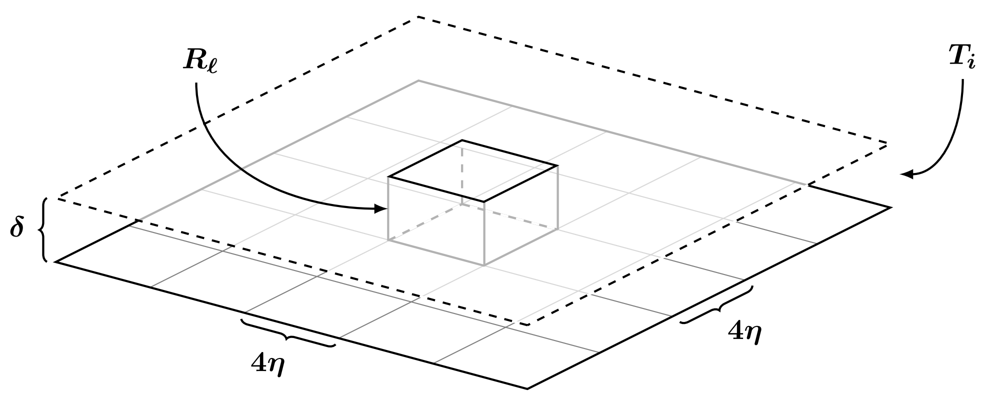

Figure 2. The "checkerboard square" R_ℓ in the case $(n,k) = (3,1)$. By choice, $\delta < \eta$, and the figure is stretched in the vertical direction for visual clarity.

$$\mathcal{H}_\infty^{\Delta-\tau}(R) \leq |R|^{\Delta-\tau} < \frac{\varepsilon}{2(2^{n-k}+1)}, \tag{3.8}$$

per our choice of d. It then follows from (3.7) and (3.8) that, for any choice of squares $R_1, ..., R_{2^{n-k}+1}$,

$$\mathcal{H}_\infty^{\Delta-\tau}\left((K \cap W_i) \setminus \bigcup_{\ell=1}^{2^{n-k}+1} R_\ell \right) > \frac{\varepsilon}{2},$$

so any cover of $(K \cap W_i) \setminus R$ by δ-balls contains $\gtrsim \delta^{\tau-\Delta}$ balls.

Now, (3.7) and (3.8) also entail that there exist distinct $R_1, ..., R_{2^{n-k}+1}$ whose intersections with $K \cap W_i$ each have positive $(\Delta - \tau)$-dimensional Hausdorff content. In particular,

$$\operatorname{card}\left\{x \in K_0 \colon B(x,\delta) \cap (K \cap W_i \cap R_p) \neq \varnothing\right\} \gtrsim \delta^{\tau-\Delta} \tag{3.9}$$

for $p = 1, ..., 2^{n-k}+1$, because $\{B(x,\delta) \colon x \in K_0\}$ is a cover of K. Necessarily, at least 2 of these squares R_p, R_q are mutually non-adjacent, so they are 4η-separated. Therefore, if $x, y \in K_0$ are such that

$$B(x,\delta) \cap (K \cap W_i \cap R_p) \neq \varnothing \quad \text{and} \quad B(y,\delta) \cap (K \cap W_i \cap R_q) \neq \varnothing,$$

then

$$\|x - y\| > 4\eta - 2\delta > 2\eta,$$

so that $x \sim_V y$. In conjunction with (3.9), this yields the estimate

$$\operatorname{card}\left\{(x,y) \in K_0^2 \colon B(x,\delta) \cap (K \cap W_i \cap R_p) \neq \varnothing,\, B(y,\delta) \cap (K \cap W_i \cap R_q) \neq \varnothing,\, x \sim_V y\right\} \gtrsim (\delta^{\tau-\Delta})^2.$$

No δ-ball intersects more than 3^k fat planes in $\mathcal{T}_V$, so we may sum the previous over all $i \in \{1, ..., j\}$ to get

$$\operatorname{card}\left\{(x,y) \in K_0^2 \colon x \sim_V y\right\} \gtrsim j(\delta^{\tau-\Delta})^2 \gtrsim \delta^{(\Delta+\tau-\gamma)+2(\tau-\Delta)} \geq \delta^{3\tau+s-2\gamma},$$

where the final inequality follows from our original hypothesis that $\Delta \geq \gamma - s$. This is the desired lower bound on the individual summands in (3.6), and multiplying by $\operatorname{card} E_0$ yields the desired bound on $\mathcal{E}$ itself:

$$\mathcal{E} \gtrsim \operatorname{card} E_0 \cdot \delta^{3\tau+s-2\gamma}.$$

[Step 4] In combination with our upper bound $\delta^{-k(n-k)+k-2\gamma'} \gtrsim \mathcal{E}$, this at last provides a concrete upper bound on $\operatorname{card} E_0$ in terms of δ, namely,

$$\operatorname{card} E_0 \lesssim \delta^{-k(n-k)+k-2\gamma'-(3\tau+s-2\gamma)} = \delta^{-k(n-k)+(k-s)-3\tau-2(\gamma'-\gamma)}.$$

Since $\gamma' > \gamma$ was arbitrary, the estimate $\operatorname{card} E_0 \lesssim \delta^{-k(n-k)+(k-s)-3\tau}$ follows at once. This holds for every δ-separated subset $E_0 \subseteq E_{\varepsilon,\tau}$, so we conclude (3.4) and, in turn, (3.3). ■

Proof of Theorem 3.1. Suppose that K is homogeneous, so that $\dim K = \overline{\dim}_B K$ and, for all $V \in \mathbf{Gr}(n,k)$, π_V is DC for K. Then Lemmas 3.3 and 3.4 combine to yield

$$\begin{aligned}\dim_P \left\{V \in \mathbf{Gr}(n,k) \colon \dim K_V \leq s\right\} &\leq \dim_P \left\{V \in \mathbf{Gr}(n,k) \colon \Delta \geq \dim K - s \ \forall \Delta \in \Delta(V)\right\} \\ &= \dim_P \left\{V \in \mathbf{Gr}(n,k) \colon \pi_V \text{ is DC for } K \text{ and } \Delta \geq \dim K - s \ \forall \Delta \in \Delta(V)\right\}\end{aligned}$$

$$\leq \dim_P \left\{ V \in \mathbf{Gr}(n,k) \colon \exists \Delta \in \Delta(V) \text{ s.t. } \Delta \geq \dim K - s \right\}$$
$$\leq k(n-k) - (k-s).$$

If instead K is self-similar with finite rotation group and if $\varepsilon > 0$, then, by [Orp12] Lemma 2.4, there exists a homogeneous set $K^\varepsilon \subseteq K$ with $\dim K^\varepsilon > \dim K - \varepsilon$. It then follows from the above that

$$\begin{aligned} \dim_P \left\{ V \in \mathbf{Gr}(n,k) \colon \dim K_V \leq s \right\} &\leq \dim_P \left\{ V \in \mathbf{Gr}(n,k) \colon \dim K_V^\varepsilon \leq s \right\} \\ &\leq k(n-k) - (k-s) \end{aligned}$$

for every $0 \leq s < \dim K - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the desired inequality must hold for all $0 \leq s < \dim K$. ■

3.3 Further directions

Orponen's 2015 paper is rife with other results that beg generalization to higher dimensions, and the same combinatorial approach taken to prove Theorem 3.1 will in all likelihood work just as well in proving these extensions. It is less clear that this will remain the case for the packing dimensional analogue of Theorem 1.15(b), or whether its statement is even true. However, there is no obvious impediment to discretizing its proof in the same way that we discretized the proof of Theorem 1.15(a) to obtain Theorem 3.1. As such, we record this as a conjecture.

Conjecture 1. *Let $K \subset \mathbb{R}^n$ be homogeneous or self-similar and let $0 < k < n$ be an integer. If $\dim K > k$, then*

$$\dim_P \{ V \in \mathbf{Gr}(n,k) \colon \dim K_V \leq s \} \leq k(n-k) - (\dim K - s)$$

for all $0 \leq s \leq k \leq \dim K < k(n-k) + s$.

The theoretical underpinning of Theorem 3.1 is Furstenberg's principle of dimension conservation. Philosophically, dimension conservation affords us partial knowledge of why the dimension of a set may have dropped upon projection onto a subspace: the dimension of the fibers accounts for a "substantial portion" of the dimension lost—the dimension did not simply "vanish"—and this concrete information enables us to deduce what is happening upstairs in $\mathbb{R}^n$ from what we see downstairs in $\mathbb{R}^k$.

A paper [FFK20] of Falconer, Fraser, and Kempton introduces a continuous spectrum of *θ-intermediate dimensions* $\dim_\theta$ that bridge the gap between Hausdorff and box dimensions, and Burrell, Falconer, and Fraser in [BFF21] were quick to acknowledge this as a tool and language for new projection theorems. Instead of using dimension conservation to go directly from the images back to the original set, the parameter θ might allow us to recover information about the packing dimension of the exceptional set from what we already know about the Hausdorff dimension of the exceptional set, i.e., from Marstrand's projection theorem. This leads me to suggest a sweeping generalization of Theorem 3.1.

Conjecture 2. *Let $K \subset \mathbb{R}^n$ be a bounded set such that the map $\theta \mapsto \overline{\dim}_\theta K$ is continuous at $\theta = 0$. Then the conclusion of Theorem 3.1 holds, and if $\dim K > k$, then the conclusion of Conjecture 1 holds.*

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